Fortran code for computations of the reflection matrix by a semi-infinite discrete random medium. Calculation examples.

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A Fortran code for fast computation of the reflection matrix of light for a semi-infinite discrete random medium in the case of oblique incidence of light on the boundary of the medium is described. It is assumed that the medium is homogeneous, isotropic and mirror symmetric and the waves propagating between the scatterers of the medium are spherical. The reflection matrix is the sum of two components, one of which corresponds to the incoherent part and is described by the vector radiative transfer equation. The second component corresponds to the coherent part and is realized in the weak localization effect. The code is designed to numerically solve both the vector radiative transfer equation and the equation for weak localization. Medium scatterers can be of arbitrary shape and composition, but in the general case it is necessary to pre-calculate the expansion coefficients of the scattering matrix by an elementary volume in series of generalized spherical functions, as well as the single scattering albedo and the scattering cross section. Several examples of reflection matrix calculation are given.

Keywords: multiple scattering; coherent backscattering; opposition effects; discrete random medium; reflection matrix

Highlights:

Electromagnetic scattering by discrete random media
Coherent backscattering
Fortran code for calculation of the reflection matrix

1. Introduction

Light scattering by random media is an open theoretical and computational problem. This problem is important for many applications, in particular for the interpretation of numerous observations of atmosphereless celestial bodies (see, e.g., [1,2]). The interpretation of these or

similar observations requires the existence of theoretical models that allow one to calculate the characteristics of light reflected by the medium, and in the general case the reflection matrix, fairly quickly and with good accuracy. At present, such models are based on the assumption that spherical waves propagate between the scatterers of the medium [3-5]. Under this assumption, the scattering characteristics of the medium are determined only by contributions of the ladder and cyclical diagrams in the diagrammatic representation of the Bethe–Salpeter equation [3-5]. The ladder diagrams correspond to the incoherent part of the reflected radiation, the cyclical diagrams correspond to the coherent part relevant to the weak localization effect. Strictly speaking, this assumption is valid only for sparse media and the corresponding models can be applied to media with a relative particle concentration up to 10% and particle sizes of the order of the wavelength of light [6]. The applicability of the e can be expanded to 15-20%, taking into account the correlation between particles by utilizing the so-called static structure factor [6-9] or by numerical modeling [10,11].

In this paper, we present a Fortran code for computing the light reflection matrix for a semi-infinite discrete random medium. In Ref. [12], a fast algorithm and Fortran code for calculating the interference component of the reflection matrix by a semi-infinite discrete random medium were presented for the case of *normal* incidence of radiation on the boundary of the medium. It was assumed that the medium is homogeneous, isotropic and mirror symmetric, and the waves propagating between the scatterers of the medium were assumed to be spherical, i.e. the scatterers are in the far zones of each other. The scattering properties of the medium are determined by the scattering matrix of the so-called elementary volume of the medium [3]. The expansion coefficients of this matrix in generalized spherical functions, as well as the single scattering albedo and scattering cross section must be known. The reflection matrix corresponding to the incoherent (diffuse) component must also be pre-calculated. Using these data, the code [12] allows to calculate the coherent (interference) component. Then the calculations provide the relative intensity and degree of linear polarization of the reflected radiation.

The present paper describes a code for computing the reflection matrix in the case of *oblique* incidence of radiation on the boundary of the medium. This code is also implemented in Fortran 90, however, unlike [12], the code is designed to numerically solve both the vector radiative transfer equation and the equation for weak localization. The code is publicly available. The solution of the vector radiative transfer was obtained using the code described in [13] and presented for free use on the website [14].

The code for the numerical solution of the equation corresponding to weak localization is based, like the code of Ref. [12], on some simplifications, which are described below. The scatterers of the medium can be of an arbitrary shape and composition. If the medium consists of identical spherical particles, it is sufficient to specify the size, refractive index of the particles and their concentration to compute the reflection matrix. If the medium consists either of polydisperse spherical particles or randomly oriented arbitrary particles, then, as for the code [12], it is necessary preliminary to calculate the coefficients of expansion of the scattering matrix by an elementary volume in series of generalized spherical functions, as well as the single scattering albedo and the scattering cross section. If correlation between scatterers is to be considered, these calculations must be made with the structure factor [7,8].

2. Basic assumptions and approximations

The code proposed in this work is designed to calculate the reflection matrix of a medium in the form of a homogeneous isotropic mirror-symmetric semi-infinite layer containing arbitrary scatterers. The geometry of radiation reflection by the medium is shown in Fig. 1. Here the incident plane wave propagates at a zenith angle ϑ_0 to the surface of the medium, vectors $\mathbf{k}_0$ and $\mathbf{k}_{sc}$ indicate the incidence direction and the scattering direction respectively. The reflection matrix is calculated in the plane of radiation incidence, i.e. in the plane formed by the normal to the surface of the medium and the direction towards the radiation source (in the *xz* plane, Fig. 1). The code calculates the elements of the reflection matrix depending on the phase angle α in the reflection regions A and B.

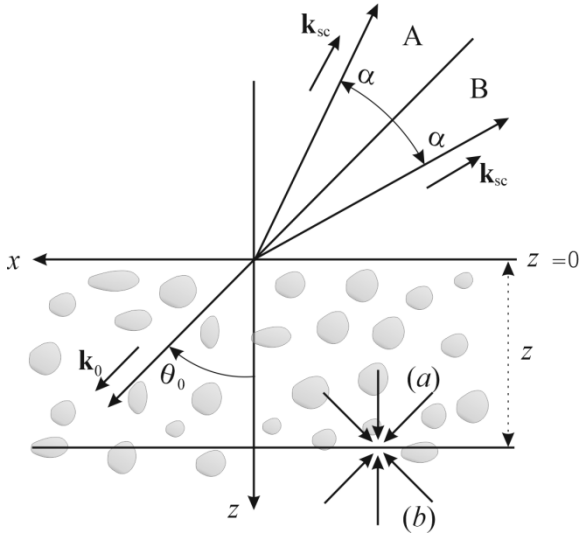

Fig.1. Scheme of radiation reflection by a medium. The radiation coming to some point at depth *z* from above (a) and below (b) is shown schematically. The other notations are explained in the text.

The assumption that spherical waves propagate between the scatterers of the medium allows one to write the reflection matrix **R** by the medium as a sum of only two terms [3,4,15]

$$\mathbf{R} = \mathbf{R}^{\mathrm{L}} + \mathbf{R}^{\mathrm{C}}. \quad (1)$$

Here the matrix $\mathbf{R}^{\mathrm{L}}$ corresponds to the incoherent (diffuse) part of the reflected radiation. In the Twersky approximation [16], this matrix is described by the vector radiative transfer equation [3]. One of the efficient and numerically accurate methods for solving this equation is described in [13] and Fortran code is published on the website [14]. This code is used with minor modifications in the code described here. The changes concern the adaptation of code [14] to Fortran 90 and the conversion of static arrays to dynamic ones.

The $\mathbf{R}^{\mathrm{C}}$ matrix corresponds to the interference part of the reflected radiation and describes the weak localization effect (or the effect of coherent backscattering) [3,4,17]. In the circular polarization basis (CP representation), the equation for this part of the reflected radiation was published in [4,17,18], and in the linear polarization basis (LP representation) in [5]. This equation is more complex than the vector radiative transfer equation and its numerically accurate solution has not yet been obtained. A certain approximate method for solving this equation was proposed in Ref. [19]. It is assumed in this approximation that the radiation coming to the point *z* from below (Fig. 1) and scattered more than twice decreases with growing the depth *z* in the layer as $\exp(-2\sigma z\mathrm{Im}(m_{eff}))$, where m_{eff} is the complex effective refractive index of the medium (see, for example, [3,20]). The parameter σ describes the rate of the decrease of the radiation characteristics with depth [12,19]. It is determined using Mishchenko's relations [1,2] relating the diagonal elements of the reflection matrix of the incoherent component to diagonal elements of weak localization component in the exactly backscattering direction. This assumption reduces a system of complex linear integral equations to a system of linear algebraic equations, the solution of which can be obtained by well-known methods. To calculate the reflection matrix of a medium consisting of polydisperse spherical particles or non-spherical particles, another approximation is used. In this approximation, some new coefficients, which have not been previously encountered in scattering theory, are replaced by well-known ones. For a medium of identical spherical particles, the code allows one to estimate the error in calculating the reflection matrix with such a replacement (see the last section). These two approximations are described in detail in [4,12,19].

3. Description of the structures of the input and output files

The *data_input.txt* input file contains the parameters needed to calculate the reflection matrix. Odd lines of the file contain a description of the input parameters and are ignored by the code. Even lines contain numeric data that define the values of the input parameters. These parameters are as follows:

– *n_case*. This parameter is an arbitrary positive integer or zero. If this parameter is 0, the *pbrf.write* file will be generated first, which will contain the Fourier components of the diffuse matrix. Then the diffuse reflection matrix will be calculated for a given angle of incidence of radiation on the medium. If the *pbrf.write* file already exists for the required parameters of the medium scatterers, and it is needed to calculate the diffuse matrix for another angle of radiation incidence on the medium (or for a different value of particle concentration), one can specify any integer instead of 0. The name of the file containing the result of the diffuse matrix calculation is determined by the specified input parameters (see examples below).

– *ng*. This parameter specifies the number of points in the Markov quadrature formula when calculating the Fourier components of the diffuse matrix. Its value must be large enough so that the interpolation of the diffuse matrix elements would be of sufficient quality in the required values of the phase angles when calculating the sum of the diffuse and coherent matrices. The code uses the spline interpolation of the diffuse matrix elements. The value *ng*=100 was sufficient in all tested cases. The Markov quadrature formula with this number of points is also used when calculating the coherent component of the reflection matrix.

– *alp0, dalp0, alp1, dalp1, alp2, dalp2, alp3*. These parameters specify the phase angle intervals. The entire phase angle interval is divided into three intervals from 0 to α_1, from α_1 to α_2, and from α_2 to α_3 (*alp0, alp1, alp2, alp3*); and each of them may have its own step of changes in α $\Delta\alpha_1$, $\Delta\alpha_2$, and $\Delta\alpha_3$ (*dalp1, dalp2, dalp3)*, respectively.

– *ksi*. It is the relative particle concentration: $\xi = 4\pi\eta a^3/3$, where a is the radius of the volume-equivalent sphere of a scatterer in the medium, and η is the concentration of scatterers. If the medium contains polydisperse scatterers, the parameter a should be averaged over the particle size distribution function.

– *thet_0.* It is the incident radiation angle in degrees (0≤*thet_0*<90).

– *kl_in*. This parameter specifies the region in which the reflection matrix will be calculated: region 'A' if *kl_in*=0; region 'B' if *kl_in*=1; both regions, 'A' and 'B' if *kl_in*=2.

– *fpos.* It is the name of an output file. The name of this file is determined by the given input parameters. An example of the formation of this file name is given below.

– *fdatcoef.* It is the name of the input file of the single-scattering matrix expansion coefficients. The structure of this file is described below.

– *fdatmat.* It is the file name of the diffuse reflection matrix (see an example below).

– *n_spher*. This parameter is an arbitrary positive integer or zero. If *n_spher*=0, the code will calculate the reflection matrix for a medium of identical spherical particles with the given parameters xv, *m_r* and *m_i* (see below). If *n_spher* is not equal to 0, the code will calculate the reflection matrix for the medium of scatterers whose single scattering matrix is specified in the *fdatcoef* file.

– xv. The size parameter of a spherical particle $xv = k_0 a$, where $k_0 = 2\pi/\lambda$ is the wave number, λ is the wavelength of the incident radiation, a is the radius of a particle.

– *m_r.* The real part of the refractive index of spherical particle with the size parameter xv.

– *m_i.* The imaginary part of the refractive index of spherical particle with the size parameter xv.

The structure of the *fdatcoef* input file is as follows. The first line of the file must contain the values of the following characteristics of an elementary volume element of the medium: the single scattering albedo $\widetilde{\omega} = q_{sca}/q_{ext}$ (where q_{sca} is the scattering efficiency factor, q_{ext} is the extinction efficiency factor, respectively); the maximal value of the expansion coefficients of the scattering matrix L; the scattering efficiency factor q_{sca}; the asymmetry parameter $< \cos\vartheta >$; and the equivalent size parameter $xv = k_0 a$. The efficiency factors should be defined with respect to the volume-equivalent particle radius a. The other lines contain the expansion coefficients of the normalized scattering matrix of the elementary volume element in generalized spherical functions in the LP basis [3,13]. The elements should be in the following succession in the line: α_1^s, $\alpha_2^s + \alpha_3^s$, $\alpha_2^s - \alpha_3^s$, α_4^s, β_1^s, and β_2^s, where α_1^s, α_2^s, α_3^s, α_4^s, β_1^s, and β_2^s are the expansion coefficients of the scattering matrix elements $a_{11}(\vartheta)$, $a_{22}(\vartheta)$, $a_{33}(\vartheta)$, $a_{44}(\vartheta)$, $a_{21}(\vartheta)$, and $a_{34}(\vartheta)$, respectively; and $0 \le s \le L$. If the medium contains polydisperse scatterers, the parameters q_{sca}, q_{ext}, a and all expansion coefficients of the scattering matrix should be averaged over the particle size distribution function.

The name of the file, that contains the reflection matrix, is generated by computer code and has the form *REFL_MATR_fpos_kl_in_thet0=thet_0.txt*. The parameters *fpos, kl_in, thet_0* are taken from the *data_input.txt* input file. The structure of the output file containing the reflection matrix is as follows. The second line of the file contains values of the following quantities: xv is the volume-equivalent radius of the scatterer of the medium; *1/(2*k₀*l_tr)*, where *l_tr* is the so called transport mean free path [3]; *sigma* is the parameter σ describes the rate of the decrease of the radiation characteristics with depth z as already mentioned above (see for details also Eq.12

in Ref.[19] or Eq.9 in Ref.[12]); *ksi* is the relative packing density of particles in the medium; *Im(m_eff)* is the imaginary part of the effective refractive index of the medium $\mathrm{Im}(m_{eff}) = 3\xi q_{ext}/(8k_0 a)$ and $\mathrm{Re}(m_{eff}) \cong 1$ (see, for example, [3,20]). We assume that $\mathrm{Re}(m_{eff}) = 1$.

The fourth line contains the values of the diagonal elements of the diffuse reflection matrix in the exact backscattering direction, the fifth line contains the values of the same elements, calculated according to Mishchenko's relations [3] using the diagonal elements of the coherent reflection matrix. Comparison of these elements allows one to estimate the accuracy of the above approximations made to calculate the coherent matrix. The seventh line contains two values of element *R_11* of the coherent matrix strictly in the direction of backscattering. The first value corresponds to that obtained by solving the equation for weak localization, the second one is obtained from the diagonal elements of the diffuse matrix using Mishchenko's relations [3]. Significant differences between these two lines indicate the need to increase the *ng* parameter in the input file *data_input.txt* to increase the accuracy of solving the weak localization equation. The next lines of the output file contain the elements of the reflection matrix, which is the sum of the diffuse and coherent matrices. The first column shows the values of the phase angle α, the next eight columns contain the values of the reflection matrix elements R_{11}, R_{12}, R_{21}, R_{22}, R_{33}, R_{44}, R_{34}, R_{43}. The tenth column contains the degree of linear polarization of the reflected radiation

$$P(\alpha) = -\frac{R_{21}(\alpha)}{R_{11}(\alpha)} \cdot 100 \tag{3}$$

and the eleventh column contains the enhancement factor [3]

$$\zeta_I(\alpha) = \frac{R_{11}^{\mathrm{L}}(\alpha) + R_{11}^{\mathrm{C}}(\alpha)}{R_{11}^{\mathrm{L}}(\alpha)} \tag{4}$$

If the sizes of the medium scatterers are very small, the range of angles, where interference is constructive, will also be small. In order to determine the angular profiles of the weak localization effect in this case, it is necessary to set small values of the parameters *alp0, dalp0, alp1, dalp1, alp2, dalp2, alp3* in the *data_input.txt* input file. The optimal choice is selection of the same values $\Delta\alpha_{1,2,3} = \Delta\alpha$. If $\Delta\alpha < 0.001$, the twelfth column will be formed in the output file, which will contain the parameter

$$q = \alpha \frac{k_0}{\eta C_{ext}}, \tag{5}$$

where C_{ext} is the extinction cross section of the medium particle.

In addition to the specified files, the code generates two more files. One of them with the name *Matr_Diffus_kl_in_thet0=thet_0.txt* contains the diffuse reflection matrix, the second one

with the name *Matr_Coher_kl_in_thet0=thet_0.txt* contains the matrix of the coherent component.

4. Examples of reflection matrices calculations

Figures 2-5 show examples of the code application for calculating reflection matrices by a medium. Fig.2 shows the enhancement factor and degree of linear polarization of the radiation reflected by a medium of identical spherical particles as a function of the phase angle when radiation is incident at an angle $\vartheta_0 = 30^0$. The input parameters for this example are: the particle size parameter $xv = 5.0$, their refractive index m=m_r+im_i =1.5+i0.0 and m=1.5+i0.1, and two particle concentrations *ksi* =0.01 and *ksi* =0.05.

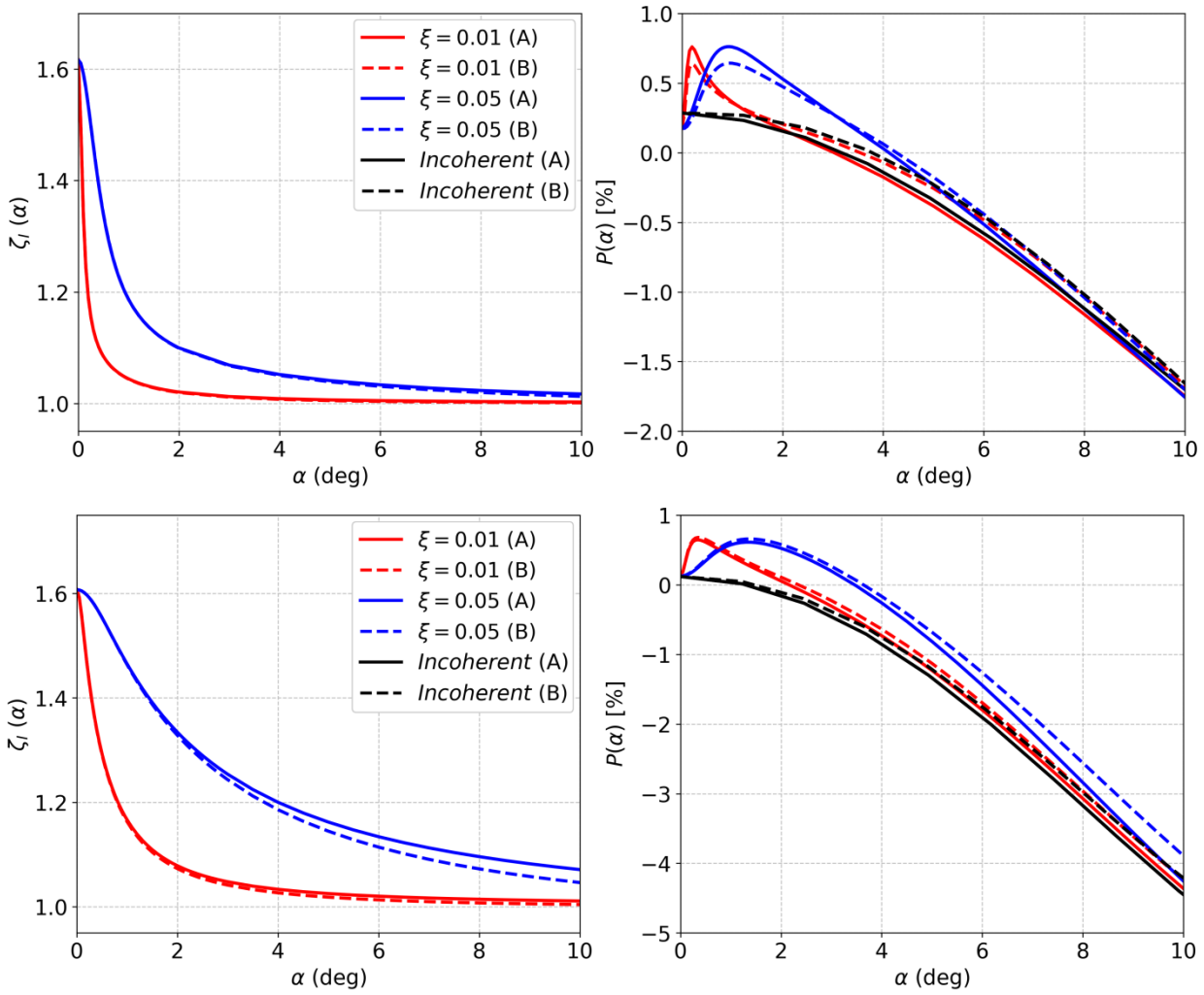

Fig.2. Dependence of the enhancement factor ζ_I and the degree of linear polarization P on the phase angle α for a medium of identical spherical particles. The upper panel corresponds to the

refractive index m=1.5+i0.0, the lower one is for m=1.5+i0.1. Scattering regions A and B are indicated in the figure, and the other parameters are specified in the text.

Figure 3 shows the enhancement factor and degree of linear polarization of the radiation reflected by a medium of randomly oriented identical prolate spheroids with the refractive index m=1.5+i0.0 and m=1.5+i0.1. The size parameter of the spheroid is $k_0\tilde{b} = 5.0$, where $\tilde{b}$ is the rotational semiaxis, and the other semiaxis is $\tilde{a} = 0.5\tilde{b}$. As in Fig. 2, the angle of radiation incidence is 30 degrees and two concentration values are accepted: *ksi* =0.01 and *ksi* =0.05. The *fdatcoef* input file for this case was generated using the T-matrix code for randomly oriented nonspherical particles [14], which made it possible to calculate the expansion coefficients of the scattering matrix by spheroids in generalized spherical functions.

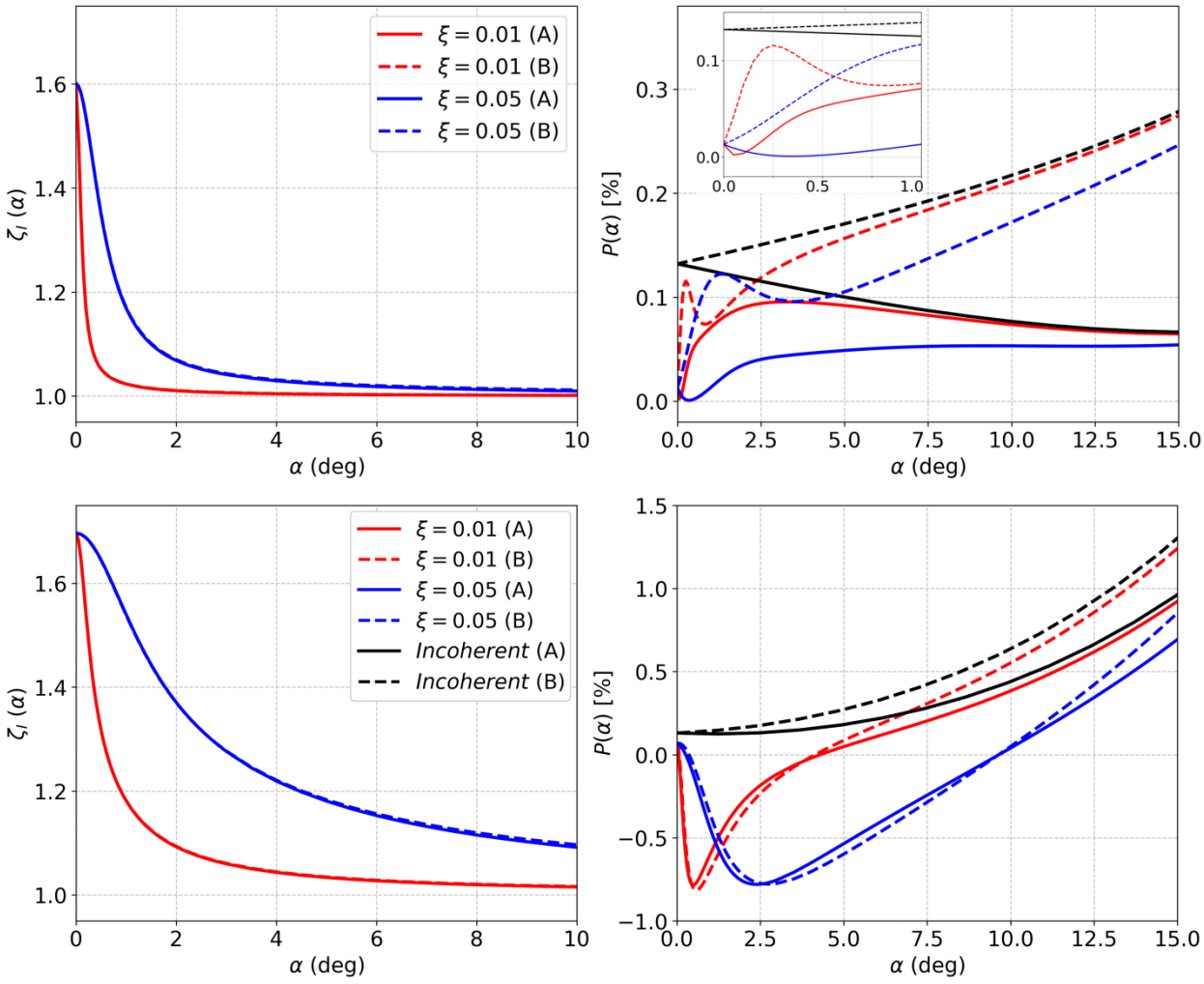

Fig.3. Dependence of the enhancement factor ζ_I and the degree of linear polarization P on the phase angle α for a medium of randomly oriented identical prolate spheroids. The upper panel

corresponds to the refractive index m=1.5+i0.0, the lower one is for m=1.5+i0.1. Areas A and B are indicated in the figure, and the other parameters are specified in the text.

The examples demonstrate the complex dependence of the reflection matrix elements on the medium parameters. In particular, the degree of linear polarization can take both positive and negative values. One can see from the figures that with increasing absorption by particles, the range of phase angles for which interference is constructive increases, which is caused by an increase in the imaginary part of the effective refractive index of the medium *Im(m_eff)*. One of the examples (Fig. 3, the lower right panel) shows a negative polarization branch similar to that observed for some atmosphereless celestial bodies [1,2]. Other examples of reflected intensity and polarization dependences by discrete random media can be found in the papers [4, 21] and references therein.

Figures 4, 5 show the dependence of the reflection matrix by the medium of identical spherical particles depending on their size parameter xv at normal incidence of radiation on the boundary of the medium. Refractive index of the medium particles is m=1.6+i0.05, and their concentration is *ksi* =0.05. These data can be considered as a dependence of the reflection matrix on the value inverse to the wavelength if size and refractive index of the scatterers are fixed. Fig. 4 also presents the values of the diagonal elements of the reflection matrix at $\alpha = 0$, obtained from the diagonal elements of the diffuse matrix using Mishchenko's relations [3]. Note that in this paper, the elements of the reflection matrix correspond to the definition of the Stokes parameters given in the book [3].

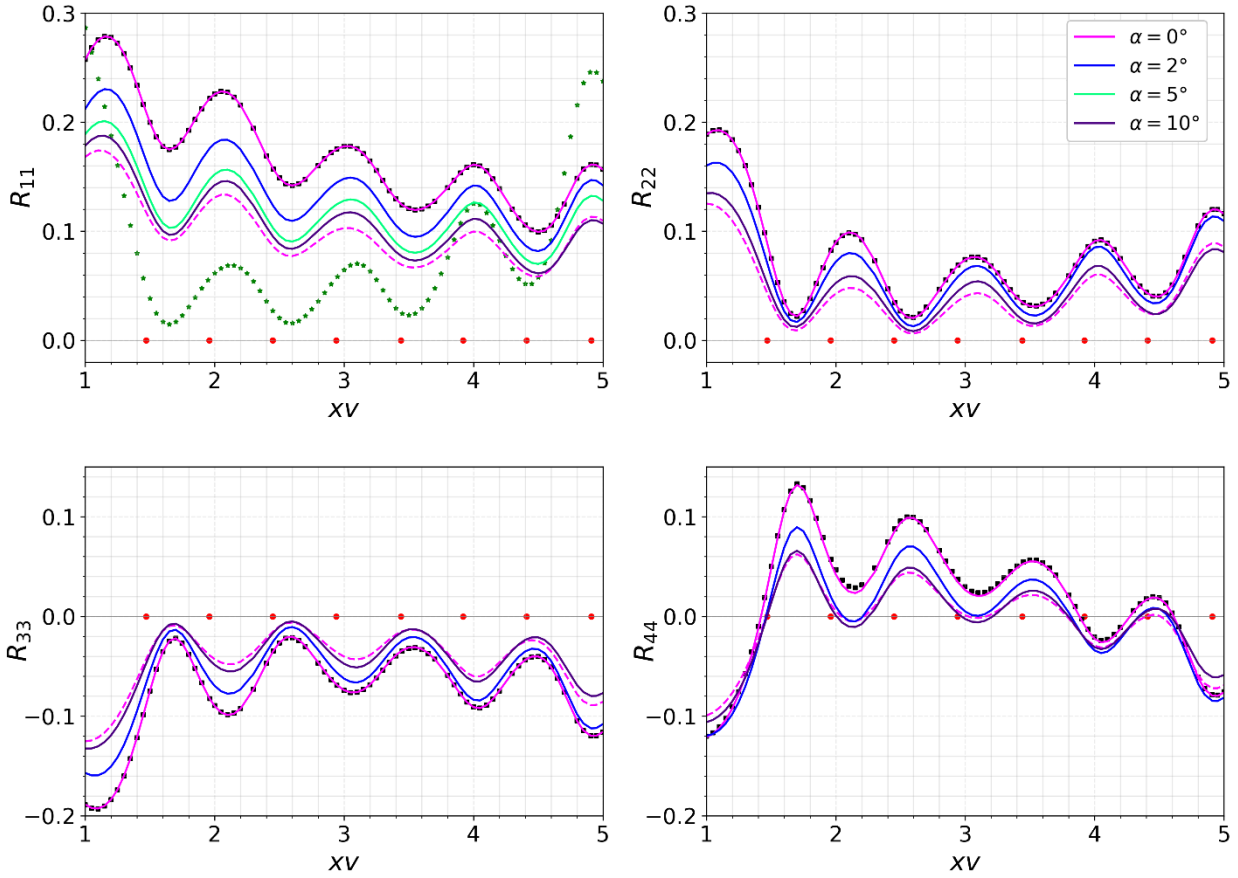

Fig.4. Diagonal elements of the reflection matrix for the medium of spherical particles as a function of size parameter xv at normal incidence of radiation on the medium. Black dots correspond to the value obtained from the diagonal elements of the diffuse matrix using Mishchenko's relations [3]. Red dots on the abscissa axis correspond to Eq.(8) at different values n. The solid colored curves represent the elements of the matrix **R** (Eq.1) at phase angles 0,2,5 and 10 degrees. The dashed curve corresponds to the elements of the matrix $\mathbf{R}^{\mathrm{L}}$ at $\alpha = 0$. These elements depend weakly on the phase angle and are therefore not shown for other angles α. The green dotted curve corresponds to the element R_{11}^{0} (Eq.6). Other parameters are specified in the text.

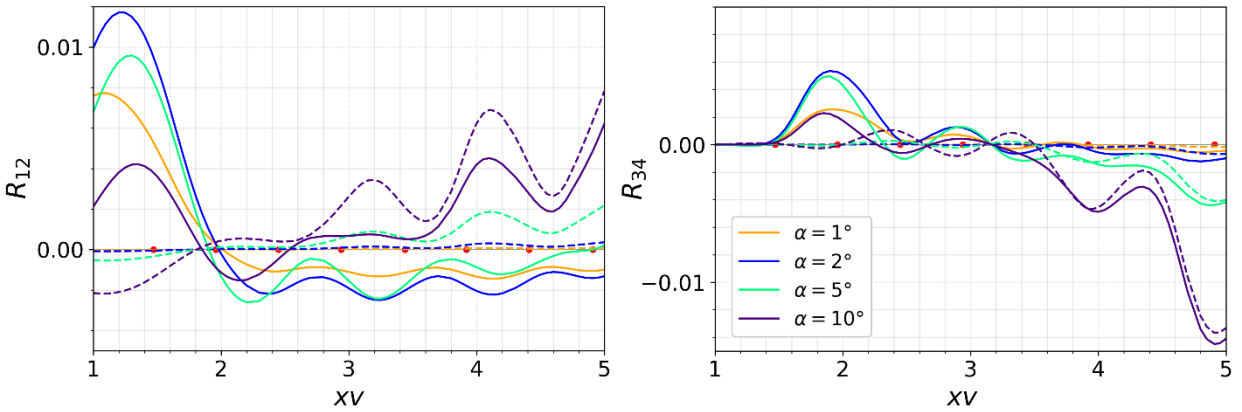

Fig.5. Elements R_{12} and R_{34} of matrices $\mathbf{R}$ and $\mathbf{R}^{\mathrm{L}}$ (Eq.1) as a function of the particle size parameter xv for phase angles 1,2,5 and 10 degrees. Solid curves correspond to elements of matrix $\mathbf{R}$, dashed ones correspond to matrix $\mathbf{R}^{\mathrm{L}}$. Red dots on the abscissa axis correspond to Eq.(8) at different values n. Elements $R_{21} \approx R_{12}$ and $R_{43} \approx -R_{34}$ are not shown here. Other parameters are specified in the text.

All matrix elements show a typical interference dependence on particle sizes with alternating minima and maxima. This dependence is explained by the resonant character of light scattering by a single particle in the backscattering direction [22]. Fig. 4 shows the dependence of the element $F_{11}(\alpha = 0)$ of the normalized scattering matrix [23] by single spherical particles on the size parameter. To combine the scales of the reflection matrix elements with the $F_{11}(\alpha = 0)$ element, this element is additionally transformed according to the formula:

$$R_{11}^{0} = \frac{F_{11}(\alpha = 0)}{3} - 0.01. \tag{6}$$

Recall that the remaining elements of the scattering matrix by spherical particles in the exact backscattering direction satisfy the equality [23]

$$F_{11}(\alpha = 0) = F_{22}(\alpha = 0) = -F_{33}(\alpha = 0) = -F_{44}(\alpha = 0) \tag{7}$$

and therefore we do not present them. The position of the minima and maxima on the curves $F_{11}(\alpha = 0)$ can be approximately determined using the formula [22]

$$x_0 = \frac{\pi n}{4\mathrm{Re}(m)}, \tag{8}$$

where $n = 3,4,\ldots$, and $\mathrm{Re}(m)$ is the real part of the refractive index of the particles. The values of x_0 at different n are shown in Fig. 4 and Fig. 5 on the abscissa axes. Note that the error of formula (8) decreases with increasing particle size or/and decreasing $\mathrm{Re}(m)$ [22].

As can be seen from Fig. 4 and Fig. 5, the positions of the minima and maxima in the elements of the reflection matrix depending on the particle sizes are determined by the dependence of the scattering matrix element $F_{11}(\alpha = 0)$. Therefore, spectral measurements of the reflection matrix allow one to estimate the particle size of the medium, or more precisely, the product of the particle size and $\mathrm{Re}(m)$ for media consisting of spherical particles.

5. Concluding remarks

The code presented in this paper makes it possible to calculate the reflection matrix by a semi-infinite discrete random medium under the assumption that spherical waves propagate between the scatterers of the medium. With this assumption, the reflection matrix is the sum of only two matrices - one of them corresponds to the diffuse component, the second one refers to the coherent component of the reflected radiation. The diffuse component is described by the classic vector radiative transfer equation, for a numerically exact solution of which the code [14] adapted for the Fortran 90 was used. The method for solving the vector radiative transfer equation is based on a Fourier series expansion in azimuth of this equation in the laboratory coordinate system (*xyz*) (Fig.1). The so-called Markov quadrature formulas [24] are used in the iterative solution of the expanded equation [13]. Then, the reflection matrix can be calculated in the laboratory coordinate system at any azimuthal angle and a polar angle corresponding to the quadrature division points.

The iterative method Bi-CGSTAB [25] is used to solve the linear system of equations corresponding to the coherent part of the reflection matrix. The code calculates the matrix of the coherent component at selected phase angles α in a coordinate system with the *z* axis along the vector $\mathbf{k}_0$. To calculate the reflection matrix containing the coherent and incoherent parts for the selected direction of the vector $\mathbf{k}_{sc}$, it is necessary to transform the matrix of the incoherent part to this coordinate system and interpolate the matrix elements to the selected phase angle. Such transformations have the simplest form if the vectors $\mathbf{k}_{sc}$ and $\mathbf{k}_0$ are located in the *xz* plane of the laboratory coordinate system. This case is most in demand at least when interpreting observations of atmosphereless bodies of the Solar System. In the special case when the vector $\mathbf{k}_{sc}$ is not located in the *xz* plane, more complex transformations are required, which significantly increase the time of calculating the reflection matrix.

The method for solving the equation for the coherent component, like the method for solving the transfer equation, is based on the expansion of the reflection matrix in generalized spherical functions, more precisely in the Wigner functions. This approach makes it possible to

calculate the coherent component in an arbitrary set of phase angles, which is important since the weak localization effect manifests itself in a very narrow range of angles near the exactly backscattering direction. This method assumes that the scattering matrix of an elementary volume of the medium is also expanded in a series of generalized spherical functions. Arbitrary particles, whose scattering matrix is averaged over orientations and other necessary parameters, in particular randomly oriented clusters of spherical particles [26], can act as an elementary volume, provided that the medium remains mirror symmetric. The time required to solve the equation for the coherent component in this method depends on the size of the medium scatterers. As the particle size increases, the number of terms in the expansion of the scattering matrix by the elementary volume of the medium in the Wigner functions increases. This leads to an increase of the system of equations, the solution of which is necessary to calculate the coherent component, and consequently to an increase in the calculation time. For example, the typical time for calculating the reflection matrix for a medium containing scatterers with sizes of the order of the incident radiation wavelength is several minutes using an ordinary PC. For a medium containing particles noticeably larger than the wavelength, the calculation time can increase to several hours.

The code uses two approximations, which are discussed in detail in the papers [4,12,19]. One of them assumes that the characteristics of the radiation arriving at point z "from below" (Fig. 1) and scattered more than twice, decrease exponentially. In the classical theory of radiation transport in a semi-infinite atmosphere a similar exponential dependence of intensity is applied in order to describe the radiation field at large optical depths [27]. For a medium of identical spherical particles only this approximation (with parameter $n_spher = 0$) is used in the code. For a medium of polydisperse particles or non-spherical particles, another approximation is applied, in which some complex coefficients that correctly describe the wave interference are replaced by well-known coefficients of the expansion of the scattering matrix in a series of generalized spherical functions [3]. This approach allows one to estimate the error of such a replacement for a medium of identical spherical particles from a comparison of the reflection matrix calculated with parameter $n_spher = 0$ and $n_spher \neq 0$ as described above. Our calculations for some parameters of the medium show that, at least for the sizes of the medium scatterers of the order and less than the wavelength, such a replacement leads to insignificant errors.

The Fortran 90 code used to perform the test calculations and some examples of reflection matrix can be downloaded as open source from website http://rian.kharkov.ua/index.php/uk/software-ua.

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CRediT authorship contribution statement

Victor P. Tishkovets: Conceptualization, Methodology, Investigation, Supervision, Software. Liudmyla A. Berdina: Methodology, Validation, Data curation, Visualization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Mishchenko MI, Rosenbush VK, Kiselev NN, Lupishko DF, Tishkovets VP, Kaydash VG, Belskaya IN, Efimov YS, Shakhovskoy NM. Polarimetric remote sensing of solar system objects. Kyiv: Akademperiodyka 2010 .

[2] Kolokolova L, Hough J, Levasseur-Regourd AC. (Editors). Polarimetry of stars and planetary systems. Cambridge: Cambridge University Press; 2015. doi:10.1017/CBO9781107358249 .

[3] Mishchenko MI, Travis LD, Lacis AA. Multiple scattering of light by particles. Radiative transfer and coherent backscattering. Cambridge: Cambridge University Press; 2006.

[4] Tishkovets VP, Petrova EV, Mishchenko MI. Scattering of electromagnetic waves by ensembles of particles and discrete random media. J Quant Spectrosc Radiat Transf 2011;112 2095–127. doi:10.1016/j.jqsrt.2011.04.010.

[5] Doicu A, Mishchenko MI. Electromagnetic scattering by discrete random media. IV: coherent backscattering. J Quant Spectrosc Radiat Transf 2019;236:106565. doi:10.1016/j.jqsrt.2019.07.008.

[6] Mishchenko MI, Goldstein DH, Chowdhary J, Lompado A. Radiative transfer theory verified by controlled laboratory experiments. Opt Lett 2013;38 3522–5. http://dx.doi.org/10.1364/OL.38.003522.

[7] Ito G, Mishchenko MI, Glotch TD. Radiative-transfer modeling of spectra of planetary regoliths using cluster-based dense packing modifications. Journal of Geophysical Research: Planets, 2018;123:1203–20. https://doi.org/10.1029/2018JE005532.

[8] Tishkovets VP, Petrova EV A cross-check of the reflectance models to be used in interpretation of jbservations of regolith-like surfaces. Front. Remote Sens, 2022;3:869527. doi: 10.3389/frsen.2022.869527.

[9] Markkanen J, Penttilä A. Extension of radiative transfer coherent backscattering RT-CB code to dense discrete random media. J Quant Spectrosc Radiat Transfer 2023;310:108733. http://dx.doi.org/10.1016/j.jqsrt.2023.108733.

[10] Muinonen K, Markkanen J, Väisänen T, Peltoniemi JI, Penttilä A. Multiple scattering in discrete random media using first-order incoherent interactions. Radio Sci 2017;52(11):1419–31. http://dx.doi.org/10.1002/2017RS006419.

[11] Muinonen K, Markkanen J, Väisänen T, Peltoniemi JI, Penttilä A. Multiple scattering of light in discrete random media using incoherent interactions. Opt Lett 2018;43(4):683-6. doi:10.1364/OL.43.000683.

[12] Tishkovets VP, Petrova EV. An algorithm and codes for fast computations of the opposition effects in a semi-infinite discrete random medium. J Quant Spectrosc Radiat Transfer 2020;255:107252. https://doi.org/10.1016/j.jqsrt.2020.107252.

[13] Mishchenko MI, Dlugach JM, Chowdhary J, Zakharova NT. Polarized bidirectional reflectance of optically thick sparse particulate layers: An efficient numerically exact radiative-transfer solution. J Quant Spectrosc Radiat Transfer 2015;156:97–108. https://doi.org/10.1016/j.jqsrt.2015.02.003.

[14] https://www.giss.nasa.gov/staff/mmishchenko/brf/

[15] Tsang L, Kong JA. Scattering of electromagnetic waves. Advanced topics. New York: Wiley; 2001.

[16] Twersky V. On propagation in random media of discrete scatterers. Proc Symp Appl Math 1964;16:84–116.

[17] Tishkovets VP. Multiple scattering of light by a layer of discrete random medium: backscattering. J Quant Spectrosc Radiat Transfer 2002;72:123–37. https://doi.org/10.1016/S0022-4073(01)00059-0

[18] Tishkovets VP, Mishchenko MI. Coherent backscattering of light by a layer of discrete random medium. J Quant Spectrosc Radiat Transfer 2004;86:161–80. https://doi.org/10.1016/S0022-4073(03)00281-4.

[19] Tishkovets VP, Mishchenko MI. Approximate calculation of coherent backscattering for semi-infinite discrete random media. J Quant Spectrosc Radiat Transfer 2009;110:139–45. https://doi.org/10.1016/j.jqsrt.2008.09.005.

[20] Bohren CF, Huffman DR. Absorption and scattering of light by small particles. New York: Wiley; 1983.

[21] Tishkovets VP, Berdina LA. Characteristics of the weak localization effect for a discrete medium of spherical particles depending on their properties and the incident radiation angle. J Quant Spectrosc Radiat Transfer 2023;310:108743. https://doi.org/10.1016/j.jqsrt.2023.108743.

[22] Tishkovets VP, Petrova EV. Spectra of light reflected by aggregate structures of submicron particles. J Quant Spectrosc Radiat Transfer 2020;252: 107116. https://doi.org/10.1016/j.jqsrt.2020.107116

[23] Mishchenko MI, Travis LD, Lacis AA. Scattering, absorption, and emission of light by small particles. Cambridge, UK: Cambridge University Press; 2002.

[24] Krylov VI. Approximate calculation of integrals. New York: Macmillan; 1962.

[25] Van der Vorst HA. Bi-CGSTAB: a fast and smoothly converging variant of Bi- CG for the solution of nonsymmetric linear systems. SIAM J Sci Stat Comput 1992;13(2) 631–44. https://doi.org/10.1137/0913035.

[26] Tishkovets VP, Petrova EV. Coherent backscattering by discrete random media composed of clusters of spherical particles. J Quant Spectrosc Radiat Transfer 2013;127: 192-206. http://dx.doi.org/10.1016/j.jqsrt.2013.05.017.

[27] Sobolev VV. Light scattering in planetary atmospheres. Oxford: Pergamon Press; 1975.

Figure captions

Fig.1. Scheme of radiation reflection by a medium. The radiation coming to some point at depth *z* from above (a) and below (b) is shown schematically. The other notations are explained in the text.

Fig.2. Dependence of the enhancement factor ζ_I and the degree of linear polarization *P* on the phase angle α for a medium of identical spherical particles. The upper panel corresponds to the refractive index *m*=1.5+i0.0, the lower one is for *m*=1.5+i0.1. Scattering regions A and B are indicated in the figure, and the other parameters are specified in the text.

Fig.3. Dependence of the enhancement factor ζ_I and the degree of linear polarization *P* on the phase angle α for a medium of randomly oriented identical prolate spheroids. The upper panel corresponds to the refractive index *m*=1.5+i0.0, the lower one is for *m*=1.5+i0.1. Areas A and B are indicated in the figure, and the other parameters are specified in the text.

Fig.4. Diagonal elements of the reflection matrix for the medium of spherical particles as a function of size parameter xv at normal incidence of radiation on the medium. Black dots correspond to the value obtained from the diagonal elements of the diffuse matrix using Mishchenko's relations [3]. Red dots on the abscissa axis correspond to Eq.(8) at different values n. The solid colored curves represent the elements of the matrix **R** (Eq.1) at phase angles 0,2,5 and 10 degrees. The dashed curve corresponds to the elements of the matrix $\mathbf{R}^{\mathrm{L}}$ at $\alpha = 0$. These elements depend weakly on the phase angle and are therefore not shown for other angles α. The green dotted curve corresponds to the element R_{11}^0 (Eq.6). Other parameters are specified in the text.

Fig.5. Elements R_{12} and R_{34} of matrices **R** and $\mathbf{R}^{\mathrm{L}}$ (Eq.1) as a function of the particle size parameter xv for phase angles 1,2,5 and 10 degrees. Solid curves correspond to elements of matrix **R**, dashed ones correspond to matrix $\mathbf{R}^{\mathrm{L}}$. Red dots on the abscissa axis correspond to Eq.(8) at different values n. Elements $R_{21} \approx R_{12}$ and $R_{43} \approx -R_{34}$ are not shown here. Other parameters are specified in the text.